4029. *Proposed by Paul Bracken.*

Suppose $a > 0$. Find the solutions of the following equation in the interval $(0, \infty)$:

$$\frac{1}{x+1} + \sum_{n=1}^{\infty} \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} = x - a.$$

We received four correct solutions and will feature two different ones.

Solution 1. We present a composite of the very similar solutions by Arkady Alt and the proposer, Paul Bracken. Another similar solution was received from Oliver Geupel.

It is clear that

$$\frac{1}{x} - \frac{1}{x+1} = \frac{1}{x(x+1)}, \quad \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)(x+2)} = \frac{2}{x(x+1)(x+2)},$$

and

$$\frac{n!}{x(x+1)(x+2)\cdots(x+n)} - \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} = \frac{(n+1)!}{x(x+1)(x+2)\cdots(x+n+1)}.$$

It therefore follows by induction that

$$\frac{1}{x} - \frac{1}{x+1} - \sum_{k=1}^{n-1} \frac{k!}{(x+1)(x+2)\dots(x+k+1)} = \frac{n!}{x(x+1)(x+2)\dots(x+n)}.$$

However, for $x > 0$,

$$\lim_{n\to\infty} \frac{n!}{x(x+1)\dots(x+n)} = 0,$$

since

$$\frac{n!}{x(x+1)(x+2)\cdots(x+n)} = \frac{1}{x\,(x+1)\left(\frac{x}{2}+1\right)\cdots\left(\frac{x}{n}+1\right)}$$

and

$$(x+1)\left(\frac{x}{2}+1\right)\cdots\left(\frac{x}{n}+1\right) > 1 + x\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

Hence the left-hand side of the original equation is given by

$$\frac{1}{x+1} + \sum_{n=1}^{\infty} \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} = \frac{1}{x}.$$

Therefore the original equation is equivalent to $x^2 - ax - 1 = 0$. This quadratic equation has the following unique solution in $(0, \infty)$:

$$x_r = \frac{1}{2}(a + \sqrt{a^2+4}).$$